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Rotational effects on the oscillation frequencies of newly born proto-neutron stars

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ABSTRACT

In this paper we study the effects of rotation on the frequencies of the quasi-normal modes of a proto-neutron star (PNS) born in a gravitational collapse during the first minute of life. Our analysis starts a few tenths of seconds after the PNS formation, when the stellar evolution can be described by a sequence of equilibrium configurations. We use the evolutionary models developed by Pons et al. (1999; 2001) that describe how a non rotating star cools down and contracts while neutrino diffusion and thermalization processes dominate the stellar dynamics. For assigned values of the evolution time, we set the star into slow rotation and integrate the equations of stellar perturbations in the Cowling approximation, both in the time domain and in the frequency domain, to find the quasi-normal mode frequencies. We study the secular instability of the g-modes, that are present in the oscillation spectrum due to the intense entropy and composition gradients that develop in the stellar interior, and we provide an estimate of the growth time of the unstable modes based on a post-Newtonian formula.

Key words: gravitational waves — stars: oscillations — stars:neutron – stars: rotation — relativity – methods: numerical –

1 INTRODUCTION

It is well known that the frequencies of quasi normal modes (QNMs) of a star depend on its internal structure, and therefore on the particular evolutionary phase the star is going

through. In a recent paper (Ferrari, Miniutti & Pons 2003), to be referred to hereafter as FMP, it has been shown that these frequencies change during the first minute of life of a proto-neutron star (PNS) born in a gravitational collapse, and that the changes are mainly due to neutrino diffusion and thermalisation processes which smooth out the high entropy gradients that develop in the stellar interior. The models used in FMP were developed in Pons et al. (1999;2001) and describe the stellar evolution in terms of a sequence of equilibrium configurations; this 'quasi-stationary' approach has been shown to become appropriate a few tenths of seconds after the bounce following stellar core collapse which gives birth to the PNS. The study carried out in FMP shows that the frequency of all QNMs of a newly born PNS are much smaller than those of the cold NS which forms at the end of the evolution. Indeed, they initially cluster in a narrow region ($\nu \in [600, 1500]$ Hz) and begin to differentiate after less than a second. Unlike in zero temperature, chemically homogeneous stars, the frequency of the fundamental mode (**f**–mode) of a PNS does not scale as the square root of the average density, though the star is cooling and contracting; in addition, due to the strong thermal gradients that caracterize the initial life of the PNS, gravity modes (**g**–modes) are also present in the oscillation spectrum, and their frequencies are much higher than those of the core **g**–modes of cold NS's (Reisenegger & Goldreich 1992; Lai 1999).

In FMP all stellar models were assumed to be non rotating; this choice was motivated by the need of isolating the effects of thermal and chemical evolution on the QNM spectrum. However, NSs are expected to be born with a significant amount of angular momentum, and the aim of this paper is to investigate how rotation modifies the picture described above. In particular, we shall explore the possibility that the **g**–modes become unstable due to Chandrasekhar-Friedman-Schutz (CFS) instability. This instability was first discovered by Chandrasekhar (1970) for the $m = 2$ bar mode of an incompressible Maclaurin spheroid, and was later shown to act in every rotating star by Friedman and Schutz (1978a,b).

The CFS instability of the fundamental mode (**f**–mode) has been studied extensively in the literature in the framework of the newtonian theory of stellar perturbations (Bardeen et. al. 1977; Clement 1979), later generalized to the post-newtonian approximation (Cutler & Lindblom 1992), and more recently for fully relativistic, fastly rotating stars (Yoshida & Eriguchi 1997; Yoshida & Rezolla 2002). These studies show that the **f**–mode instability acts at very high stellar rotation rates, comparable to the break-up velocity limit of the star. It was also found that, unless the temperature is very low, viscous dissipation mechanisms tend to stabilise the **f**–mode instability. It should be stressed that, except that in Morsink, Stergioulas

& Blattning (1999) where a more realistic equation of state (EOS) has been considered, all these studies use a one parameter, polytropic EOS. The **r**–mode instability has also extensively been studied in recent years after Andersson (1998) pointed out that it is generic for every rotating star, and since the coupling of the **r**–mode with the current multipoles is strong, it was proposed that this instability plays an important role in nascent neutron stars. In principle, all quasinormal modes of a rotating star are unstable for appropriate values of the stellar angular velocity Ω, and of the spherical harmonic index m, but the more interesting ones are those for which the instability sets in for a small value of Ω. In this respect, it should be noted that since in the no rotation limit the **g**–modes have frequencies lower than the **f**–mode frequency, they may become unstable for relatively small values of the angular velocity. The **g**–mode instability has been studied only for zero temperature stars using the newtonian theory of stellar perturbations in the Cowling approximation. This study regarded a particular class of **g**–modes, for which the buoyancy is provided by the gradient of proton to neutron ratio in the interior of the star (Lai 1999). In this paper we study the onset of the CFS instability of the lowest **g**–modes of a newly born, hot proto-neutron star using, as mentioned above, evolutionary models that take into account the physical processes occurring in the early life of the star. We use the relativistic theory of stellar perturbations for a slowly rotating star in the Cowling approximation, which is known to reproduce with a good accuracy the **g**–mode frequencies because the gravitational perturbation induced by **g**–modes is much smaller than that associated to the **f**–mode.

The paper is structured as follows. In section 2 we write the equations that describe a perturbed, slowly rotating, relativistic star both in the time and in the frequency domain in the Cowling approximation. In section 3 we present and discuss the numerical results both for the fundamental mode and for the lowest **g**–modes; we estimate the growth time of the unstable **g**–modes using post-newtonian formulae and draw the conclusions of our study.

2 FORMULATION OF THE PROBLEM

2.1 The background model

We consider a relativistic star in uniform rotation with an angular velocity Ω so slow that the distortion of its figure from spherical symmetry is of order Ω^2, and can be ignored. We expand all equations with respect the parameter $\varepsilon = \Omega/\Omega_K$, where $\Omega_K = \sqrt{\frac{M}{R^3}}$, and retain

only first order terms $\mathcal{O}(\varepsilon)$. On these assumptions, the metric can be written as (Hartle 1976)

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) - 2\varepsilon\omega(r)\sin^2\theta dt d\phi. \tag{1}$$

The function $\omega(r)$ satisfies a second order linear equation

$$\varpi_{,r,r} + \frac{4}{r}\varpi_{,r} - (\lambda+\nu)_{,r}\left(\varpi_{,r} + \frac{4}{r}\varpi\right) = 0, \tag{2}$$

where we have defined

$$\varpi = \Omega - \omega(r). \tag{3}$$

In the vacuum outside the star, λ and ν reduce to the Schwarzschild functions, and the solution of eq. (2) can be written as

$$\varpi = \Omega - 2Jr^{-3}, \tag{4}$$

where J is the angular momentum of the star. The star is assumed to be composed by a perfect fluid, whose energy momentum tensor is

$$T_{\mu\nu} = (p+\rho)\,u_\mu u_\nu + p g_{\mu\nu}, \tag{5}$$

with pressure p, energy density ρ and four-velocity components $u^\mu = [e^{-\nu}, 0, 0, \Omega e^{-\nu}]$. The metric functions $\nu(r), \lambda(r)$ are found by solving Einstein's equations for a spherically symmetric, non rotating star, which couple the metric components to the fluid variables. As already mentioned in the introduction we shall use as a background the models of evolving proto-neutron stars developed in Pons et al. (1999;2001) and used in FMP; we shall choose different values of the evolution time starting from $t = 0.5$ s after the formation of the proto-neutron star, when the quasi–stationary description is appropriate to represent the stellar evolution.

2.2 The perturbed equations in the Cowling approximation

In this section we shall briefly outline the equations that describe the perturbations of a slowly rotating star up to first order in the rotation parameter ε. We shall write these equations both in the time domain and in the frequency domain, because we have used both approaches to find the mode frequencies. In both cases we shall assume the oscillations to be adiabatic, so that the relation between the Eulerian perturbation of the pressure, δp, and of the energy density, $\delta\rho$, is given by

$$\delta p = \frac{\Gamma_1 p}{p+\rho}\delta\rho + p'\xi^r\left(\frac{\Gamma_1}{\Gamma} - 1\right), \tag{6}$$

where ξ^r is the radial component of the Lagrangian displacement ξ^μ, and Γ_1 and Γ are

$$\Gamma_1 = \frac{\rho+p}{p}\left(\frac{\partial p}{\partial\rho}\right)_{s,Y_L}, \qquad \Gamma = \frac{\rho+p}{p}\,\frac{p'}{\rho'}\;; \tag{7}$$

a prime indicates differentiation with respect to r, and Y_L is the lepton fraction. In the following equations we shall also use the speed of sound C_s given by

$$C_s^2 = \left(\frac{\partial p}{\partial\rho}\right)_{s,Y_L} = \frac{\Gamma_1}{\Gamma}\frac{p'}{\rho'}. \tag{8}$$

The complete set of the perturbed Einstein equations has been derived using the BCL gauge (Battiston, Cazzola & Lucaroni 1971) in Ruoff, Stavridis & Kokkotas (2002). The Cowling limit of these equations was studied in Ruoff, Stavridis & Kokkotas (2003), (RSK) for polytropic relativistic equations of state. In this work we will use the Cowling approximation, i.e. we shall neglect the contribution of the gravitational perturbations. In this approach, we need to consider only the fluid perturbations, i.e. the three components of the velocity perturbations δu_i, the perturbation of the energy density $\delta\rho$ and the radial component of the displacement vector ξ^μ. The perturbation of the pressure is related to $\delta\rho$ through the adiabatic condition (6). We expand the fluid perturbations as

$$\begin{aligned}
\delta u_r &= -e^\nu \sum_{l,m} u_1^{lm} Y_{lm}, \qquad (9)\\
\delta u_\theta &= -e^\nu \sum_{l,m}\left[\tilde{u}_2^{lm}\partial_\theta Y_{lm} - \tilde{u}_3^{lm}\frac{\partial_\phi Y_{lm}}{\sin\theta}\right],\\
\delta u_\phi &= -e^\nu \sum_{l,m}\left[\tilde{u}_2^{lm}\partial_\phi Y_{lm} + \tilde{u}_3^{lm}\sin\theta\partial_\theta Y_{lm}\right],\\
\delta\rho &= \sum_{l,m}\frac{(p+\rho)^2}{\Gamma_1 p}\left(H^{lm}-\xi^{lm}\right)Y_{lm},\\
\xi^r &= \left[\nu'\left(1-\frac{\Gamma_1}{\Gamma}\right)\right]^{-1}\sum_{l,m}\xi^{lm}Y_{lm},
\end{aligned}$$

where $H = \delta p/(p+\rho)$ is the enthalpy.

2.2.1 Perturbed equations in the time domain

The final set of perturbed equations in the time domain is

$$\begin{aligned}
(\partial_t + \mathrm{i}m\Omega)\,H &= e^{2\nu-2\lambda}\left\{C_s^2\left[u_1' + \left(2\nu'-\lambda'+\frac{2}{r}\right)u_1 - e^{2\lambda}\frac{\Lambda}{r^2}u_2 + 2\mathrm{i}m\varpi e^{2\lambda-2\nu}H\right] - \nu' u_1\right\},\\
(\partial_t + \mathrm{i}m\Omega)\,u_1 &= H' + \frac{p'}{\Gamma_1 p}\left[\left(\frac{\Gamma_1}{\Gamma}-1\right)H + \xi\right] - B\left(\mathrm{i}mu_2 + \mathcal{L}_1^{\pm1}u_3\right),\\
(\partial_t + \mathrm{i}m\Omega)\,u_2 &= H + 2\frac{\varpi}{\Lambda}\left(\mathrm{i}mu_2 + \mathcal{L}_3^{\pm1}u_3\right) - \frac{\mathrm{i}mr^2}{\Lambda}e^{-2\lambda}Bu_1, \qquad (10)
\end{aligned}$$

$$
\begin{aligned}
(\partial_t + \mathrm{i}m\Omega)\, u_3 &= 2\frac{\varpi}{\Lambda}\left(\mathrm{i}mu_3 - \mathcal{L}_3^{\pm1}u_2\right) + \frac{r^2}{\Lambda}e^{-2\lambda}B\mathcal{L}_2^{\pm1}u_1,\\
(\partial_t + \mathrm{i}m\Omega)\,\xi &= \nu'\left(\frac{\Gamma_1}{\Gamma} - 1\right)e^{2\nu-2\lambda}u_1.
\end{aligned}
$$

where:

$$
\begin{aligned}
\Lambda &= l(l+1), \qquad Q_{lm} := \sqrt{\frac{(l-m)(l+m)}{(2l-1)(2l+1)}}, \qquad (11)\\
B &= \omega' + 2\varpi\left(\nu' - \frac{1}{r}\right).
\end{aligned}
$$

In eqs. (10) we have omitted the indices lm in the perturbed variables and we have introduced the new quantities

$$
\begin{aligned}
u_2 &:= \tilde{u}_2 + \mathrm{i}m\frac{\varpi r^2}{\Lambda}e^{-2\nu}H,\\
u_3 &:= \tilde{u}_3 - \frac{\varpi r^2}{\Lambda}e^{-2\nu}\mathcal{L}_2^{\pm1}H.
\end{aligned}
$$

The operators $\mathcal{L}_1^{\pm1}$, $\mathcal{L}_2^{\pm1}$, and $\mathcal{L}_3^{\pm1}$ are the same as in RSK and are defined by their action on a perturbation variable P^{lm}

$$
\begin{aligned}
\mathcal{L}_1^{\pm1}P^{lm} &= (l-1)Q_{lm}P^{l-1m} - (l+2)Q_{l+1m}P^{l+1m}, \qquad (12)\\
\mathcal{L}_2^{\pm1}P^{lm} &= -(l+1)Q_{lm}P^{l-1m} + lQ_{l+1m}P^{l+1m},\\
\mathcal{L}_3^{\pm1}P^{lm} &= (l-1)(l+1)Q_{lm}P^{l-1m} + l(l+2)Q_{l+1m}P^{l+1m}.
\end{aligned}
$$

We must stress here that the above system is an infinite coupled system of differential equations ranging from $l = m$ to $l = \infty$ and that, in order to solve it numerically, we need to truncate the couplings to a finite value of $l_{\max}$.

In this work we want to study how the **f**-and **g**-modes, that have polar parity in the non rotating limit, are affected by rotation. When the star rotates, the equations that describe a polar perturbation with harmonic index l acquire rotational corrections with polar parity and index l, (for instance the terms $\mathrm{i}m\Omega H$ and $2\mathrm{i}m\varpi e^{2\lambda-2\nu}H$ in the first of eqs. (10)) and with axial parity and index $l \pm 1$ (for instance the term $\mathcal{L}_1^{\pm1}u_3$ in the second of eqs. (10)).

Since it has been shown that the polar rotational corrections are dominant for the modes under investigation (Kojima 1997), in our study we shall neglect the ±1 axial rotational corrections. Thus, the modes we are considering are those indicated in (Lockitch, Andersson & Friedman2001) as the "polar led hybrid modes". We choose $l_{\max} = 2$ and $m = 2$. We have checked that considering higher values of $l_{\max}$ would change the results by less than 1%.

Eqs. (10) have been numerically integrated giving an initial gaussian pulse at the enthalpy

variable H. The frequencies of the modes are identified by looking at the peaks of the fast fourier transfom (FFT) of the outcoming signal.

2.2.2 Perturbed equations in the frequency domain

By replacing in Eqs. (10) all time derivatives by iσ and letting $H \to \mathrm{i}H$, $u_3 \to \mathrm{i}u_3$ and $\xi^r \to \mathrm{i}\xi^r$, we easily obtain the real valued set of equations describing the eigenvalue problem in the frequency domain. We have two ODEs for H and u_1 and three algebraic relations for u_2, u_3 and ξ^r. The relation for ξ^r, which follows from the last of eqs. (10) is particularly simple and can be used to eliminate that variable from the system. The result is

$$\begin{aligned}
H' &= (\sigma + m\Omega)\, u_1 - \frac{p'}{\Gamma_1 p}\left(\frac{\Gamma_1}{\Gamma} - 1\right)\left[H - (\sigma + m\Omega)^{-1}\nu' e^{2\nu - 2\lambda} u_1\right] + B\left(m u_2 + \mathcal{L}_1^{\pm 1} u_3\right), \\
u_1' &= -\left(2\nu' - \lambda' + \frac{2}{r}\right) u_1 + 2m\varpi e^{2\lambda - 2\nu} H + e^{2\lambda}\frac{\Lambda}{r^2} u_2 - C_s^{-2}\Big[(\sigma + m\Omega)\, e^{2\lambda - 2\nu} H - \nu' u_1\Big], \\
u_2 &= \Sigma^{-1}\Big[H - \frac{m r^2}{\Lambda} e^{-2\lambda} B u_1 + \frac{2\varpi}{\Lambda}\mathcal{L}_3^{\pm 1} u_3\Big], \\
u_3 &= -\Sigma^{-1}\Big[\frac{r^2}{\Lambda} e^{-2\lambda} B \mathcal{L}_2^{\pm 1} u_1 - \frac{2\varpi}{\Lambda}\mathcal{L}_3^{\pm 1} u_2\Big],
\end{aligned} \tag{13}$$

where we have defined

$$\Sigma := \sigma + m\Omega - \frac{2m\varpi}{\Lambda}. \tag{14}$$

An inspection of eqs. (13) shows that they become singular when $\Sigma := 0$. For any assigned value of l, m, Ω and σ, this may happen inside the star in a certain domain of the radial coordinate r which would depend on the values of the function ϖ. This occurrence would generate the so-called continuous spectrum. As explained in RSK, for fixed values of the compactness of the star M/R, the frequency region were the continuous spectrum extends pratically depends on the values of ϖ at the center and at the surface of the star and on the number of maximum couplings $l_{\max}$ that is considered. By following the procedure explained in RSK (section 2.3), it is easy to show that in the non-axisymmetric case ($m \neq 0$) and for $l_{\max} = 2$ the continuous spectrum extends to the following region

$$2\left(\frac{\varpi_c}{3} - \Omega\right) \leq \sigma \leq 2\left(\frac{\varpi_s}{3} - \Omega\right). \tag{15}$$

Assuming $l = m = 2$, Eqs. (13) can be written in the following simplified form

$$\begin{aligned}
H' &= \left[\frac{mB}{\Sigma} - \frac{p'}{\Gamma_1 p}\left(\frac{\Gamma_1}{\Gamma} - 1\right)\right] H + \left[\sigma + m\Omega + (\sigma + m\Omega)^{-1}\frac{p'}{\Gamma_1 p}\left(\frac{\Gamma_1}{\Gamma} - 1\right)\nu' e^{2\nu - 2\lambda}\right] u_1 \quad (16) \\
u_1' &= \left[-2\nu' + \lambda' - \frac{2}{r} + C_s^{-2}\nu' - \frac{mB}{\Sigma}\right] u_1 + \left[e^{2\lambda - 2\nu}\Big(2m\varpi - C_s^{-2}(\sigma + m\Omega)\Big) + \frac{\Lambda e^{2\lambda}}{r^2 \Sigma}\right] H
\end{aligned}$$

where we have used the third of eqs. (13) to eliminate u_2. To find the mode frequencies, we integrate these two ODEs by imposing that the variables have a regular behaviour near the center, i.e.

$$H \sim r^l, \qquad u_1 \sim r^{l-1}, \tag{17}$$

and select those frequencies for which the Lagrangian perturbation of the pressure vanishes at the surface, i.e.

$$\Delta p = (\sigma + m\Omega)H(R) - \nu'(R)e^{4\nu(R)}u_1(R) = 0. \tag{18}$$

3 RESULTS

In order to find the frequency of the quasi normal modes, we have numerically integrated the perturbed equations both in the time and in the frequency domain for different values of the evolution time t_{ev}, and for selected values of the rotation parameter ε. We choose the rotation rate to vary within $0 \le \varepsilon \le 0.4$ because from preliminary calculations we find that for the models under consideration the mass shedding limit does not exceed $\varepsilon = 0.4 - 0.5$.

We consider the evolutionary model labelled as model A in FMP, in which the equation of state of baryonic matter is a finite-temperature, field-theoretical model solved at the mean field level. Electrons and muons are included in the models as non interacting particles, being the contribution due to their interactions much smaller than that of the free Fermi gas, and neutrino transport is treated using the diffusion approximation. The evolution time interval we consider covers the first minute of life of the proto-neutron star, from $t_{ev} = 0.5\ s$ to $t_{ev} = 40\ s$, when processes related to neutrino diffusion and thermalization become negligible. The gravitational mass of the star at $t_{ev} = 0.2\ s$ is $M = 1.56\ M_\odot$, and at $t_{ev} = 40\ s$ becomes $M = 1.46\ M_\odot$. The difference in gravitational mass between the initial and final configuration is radiated away by neutrinos during the PNS evolution. The radius of the initial configuration is $R = 23.7$ km and reduces to $R = 12.8$ km at $t_{ev} = 40\ s$.

For $t_{ev} \lesssim 20\ s$ the stellar models are convectively unstable, and the code which integrates the perturbed equations in the time domain explodes after some time, which is too short to accurately calculate the low frequencies of the **g**–modes. Conversely, the code which integrates the equations in the frequency domain is well behaved even when convective instability is present, and therefore for $t_{ev} < 20\ s$ we use the frequency domain approach. After that time both methods can be applied and the results agree better than 5%.

It is worth stressing here once more that the perturbed equations in the frequency domain present a singular structure which makes impossible their numerical integration in the continuous spectrum region. However, for the stellar models we use and for the mode frequencies we are interested in, we find that the continuous spectrum lays in the negative frequency range.

The main results of this work are summarized in figures 1 and 2, where we plot the frequencies of the **f**-, $\mathbf{g}_1$- and $\mathbf{g}_2$- modes as a function of the rotation parameter $\varepsilon = \Omega/\Omega_K$, for different values of the evolution time in the more interesting phases of the cooling process.

It should be reminded that the onset of the CFS instability is signaled by the vanishing of the mode frequency for some value of the angular velocity (neutral point). From figure 1 and 2 we see that while the **f**–mode does not become unstable during the first minute of the PNS life, both the $\mathbf{g}_1$- and the $\mathbf{g}_2$- modes do become unstable. The $\mathbf{g}_1$- frequency remains positive during the first second, but at later times vanishes for very low values of ε. For instance, at $t_{ev} = 3\ s$ it crosses the zero axis for $\Omega = 0.17\ \Omega_K$, even though its value for the corresponding nonrotating star is still quite high, $\nu_{g_1} = 486$ Hz. The behaviour of the $\mathbf{g}_2$- mode is similar, but being the frequency lower the instability sets in at lower rotation rates. At later times, the **g**-modes frequencies decrease, reach a minimum for $t_{ev} = 12\ s$ and then slightly increase. This behaviour can be attributed to the fact that during the first 10-12 seconds the dynamical evolution of the star is dominated by strong entropy gradients that progressively smooth out. After about 12 s the entropy has become nearly constant throughout the star and **g**-modes due to composition gradients take over.

It should be mentioned that in FMP we studied also a second model of evolving proto-neutron star, labelled as model B. The main difference between the two models is that model A has an EOS softer than model B, and that at some point of the evolution a quark core forms in the interior of model B. We have integrated the perturbed equations (10) and (16) also for model B, finding results entirely similar to those described above for model A; this indicates that that the quark core that develops at some point of the evolution in model B does not affect the overall properties of the modes in a relevant way.

3.1 The growth time of the unstable modes

A mode instability is relevant if its growth time is sufficiently small with respect to the timescales typical of the stellar dynamics, i.e. if the instability has sufficient time to grow

before other processes damp it out or the structure of the evolving star changes. In this section we will give an "order of magnitude" estimate of the growth time of the **g**–modes for the hot proto-neutron stars under investigation. Following Lai (1999), we shall evaluate the mode energy in the rotating frame using the expression given in Friedman & Schutz (1978a)

$$E = \frac{1}{2}\int\left[\rho\,(\sigma+m\Omega)^2\,\vec{\xi}^*\;\cdot\;\vec{\xi}+\left(\frac{\delta p}{\rho}-\delta\Phi\right)\delta\rho^*+\left(\vec{\nabla}\;\cdot\;\vec{\xi}\right)\vec{\xi}^*\;\cdot\;\left(\vec{\nabla}p-C_s^2\vec{\nabla}\rho\right)\right]d^3x, \qquad (19)$$

and compute the growth time associated to the dissipative process we are considering, i.e. the gravitational radiation reaction, using the expression

$$\frac{1}{\tau_{gr}} = -\frac{1}{2E}\frac{dE}{dt}. \qquad (20)$$

It is useful to remind that the relation between the Lagrangian displacement and the four-velocity of a perturbed fluid element is

$$\delta u^k = \mathrm{i}(\sigma+m\Omega)e^{-\nu}\,\xi^k, \qquad k=1,3. \qquad (21)$$

Since we are working in the Cowling approximation, the term $\delta\Phi$ in eq. (19) will be neglected. The energy loss due to gravitational waves can be calculated from the multipole radiation formula of Lindblom et. al. (1998) given by

$$\left(\frac{dE}{dt}\right)_{\rm gr} = -\sigma\,(\sigma+m\Omega)\sum_{l\geq 2}N_l\sigma^{2l}\left(|\delta D_{lm}|^2+|\delta J_{lm}|^2\right), \qquad (22)$$

where the coupling constant N_l is given by

$$N_l = \frac{4\pi G}{c^{2l+1}}\frac{(l+1)(l+2)}{l(l-1)[(2l+1)!!]^2}, \qquad (23)$$

and δD_{lm} and δJ_{lm} are the mass and current multipoles of the perturbed fluid. The current multipoles δJ_{lm} are associated with the axial spherical harmonics; since we are interested in the **g**-modes, for which the effect of the coupling between polar and axial perturbations is negligible (indeed we did not include it in the perturbed equations), we shall neglect the δJ_{lm} contribution to the gravitational luminosity. The mass multipoles can be evaluated from the following integral expression

$$\delta D_{lm} = \int r^l\,\delta\rho\,Y^*_{lm}d^3x. \qquad (24)$$

The growth times of the unstable $\mathbf{g}_1$- modes shown in figure 2 are summarized in Table 1. From these results we see that the growth time appears to be orders of magnitude larger than the timescale on which the star evolves, which is of the order of tens of seconds. Although we are aware that the estimate based on the newtonian expressions (19) and (24) is a quite crude one, the growth time is so much larger than the evolutionary timescale that it is

reasonable to conclude that the CFS instability of the lowest **g**–mode is unlikely to play any relevant role in the early evolution of proto-neutron stars. Similar conclusions can be drawn for the fundamental mode and for higher order **g**-modes.

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Table 1. Growth times for unstable g_1 mode of Model A for $t_{ev} = 3\ s$, $t_{ev} = 10\ s$, $t_{ev} = 12\ s$, and $t_{ev} = 40\ s$

ε	ν (Hz)	τ_{gr} (s)
0.1	200	...
0.2	-90	-1.5 10^9
0.3	-683	-2 10^6
0.4	-789	-2.2 10^4

ε	ν (Hz)	τ_{gr} (s)
0.1	-150	-8.3 10^9
0.2	-435	-6.5 10^6
0.3	-683	-4 10^5
0.4	-910	-2.4 10^3

ε	ν (Hz)	τ_{gr} (s)
0.1	-160	-4.6 10^7
0.2	-475	-1.7 10^6
0.3	-760	-3.4 10^5
0.4	-1020	-2.7 10^5

ε	ν (Hz)	τ_{gr} (s)
0.1	-70	-3.7 10^{10}
0.2	-430	-1.3 10^9
0.3	-900	-3.6 10^6
0.4	-1250	-1.3 10^5

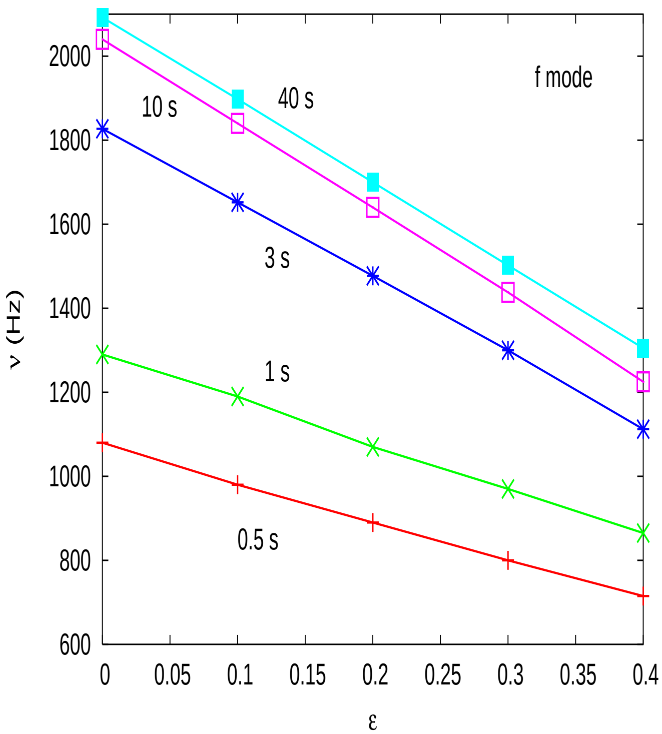

Figure 1. The frequency of the fundamental mode of the evolving proto-neutron star is plotted as a function of the rotational parameter $\varepsilon = \Omega/\Omega_K$, for assigned values of the time elapsed from the gravitational collapse. We see that as the time increases, the frequency increases and tends to that of the cold neutron star which forms at the end of the evolutionary process. Reminding that the onset of the CFS instability occurs when the mode frequency becomes zero, we see that the **f**–mode would become unstable only for extremely high values of the rotational parameter, as it is for cold stars.

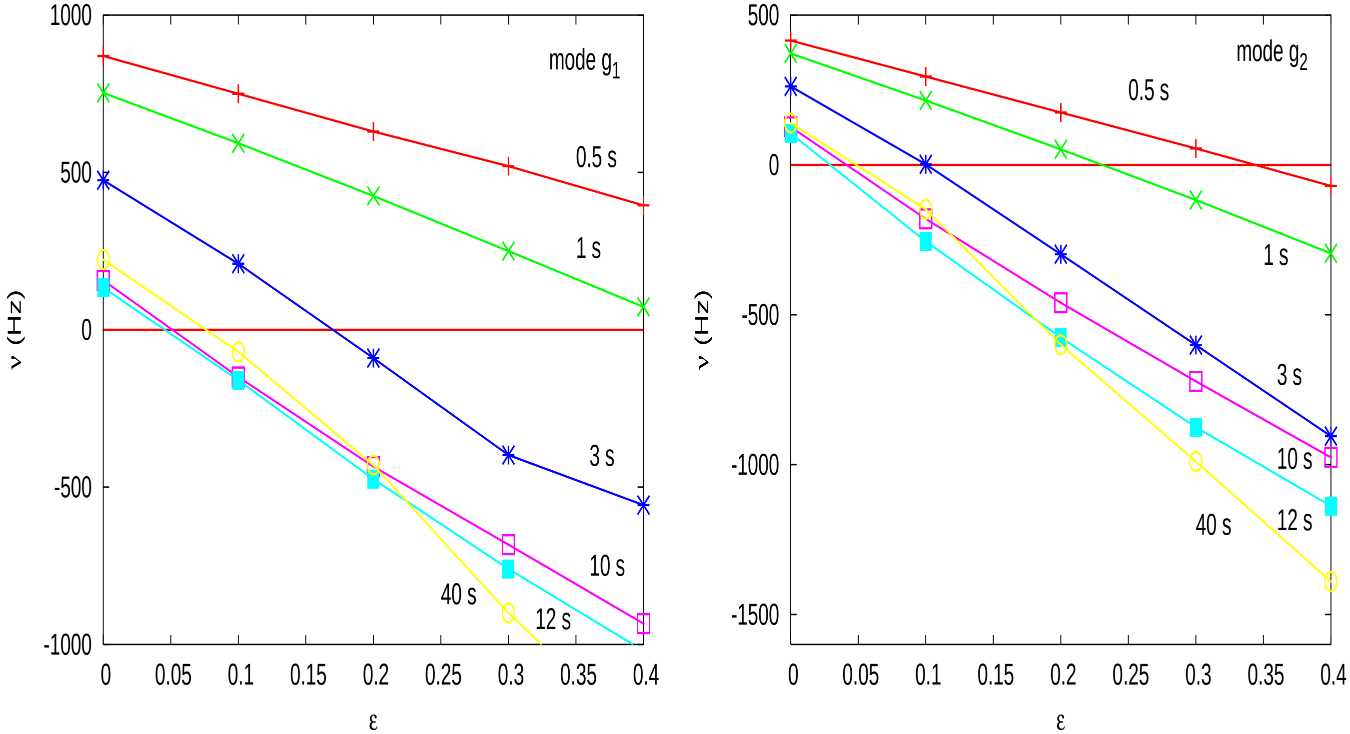

Figure 2. The frequency of the $\mathbf{g}_1$- and $\mathbf{g}_2$- modes of the proto-neutron star are plotted, as in figure 1, as functions of the rotational parameter for assigned values of the time elapsed from the gravitational collapse. Unlike the **f**–mode, as the time increases the frequency of the **g**–modes decreases, reaches a minimum at about $t_{ev} = 12$ and then slightly increases (see text). We see that for both modes the CFS instability sets in at values of the rotational parameter much lower that that needed for the **f**–mode.